Journal of Statistical Physics, Vol. 59, Nos. 5/6, 1990

Infinite-Order Phase Transition in a Classical Spin System

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Received September 18, 1989

For an exactly soluble classical spin model with long-range inhomogeneous coupling it is proved that in the absence of external magnetic field the free energy is a C^∞ function of the temperature at the critical point.

KEY WORDS: Inhomogeneous spin model; phase transition; critical behavior; exactly soluble model.

1. INTRODUCTION

Recently, various one-dimensional models with inhomogeneous interaction have been introduced as simple examples of a phase transition. Such results have been mainly concerned with nearest neighbor potentials and unbounded coupling.[1-4] However, inhomogeneous spin systems also provide the means of studying the influence of very long-range potentials on the critical behavior.

The object of this paper is an exactly soluble classical spin model with an inhomogeneous coupling decreasing roughly as $|i-j|^{-1}$ (such a decay would correspond to the boundary case $c=2$ of the hierarchical model[5]). This interaction results in a rather unusual critical behavior. We consider a system of classical spins $\{\sigma_i\}_{i\in\mathbb{N}}$, $\sigma_i=\pm1$, in the presence of a magnetic field $h\geqslant 0$. The Hamiltonian is defined by

$$\mathscr{H}_n \equiv \mathscr{H}_n(\{\sigma_i\}_{1\leqslant i\leqslant n}; h) = -\sum_{1\leqslant i\leqslant j\leqslant n} j^{-1}\sigma_i\sigma_j - h\sum_{1\leqslant i\leqslant n}\sigma_i \quad (1.1a)$$

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This result can be obtained by standard methods; we give an intuitive picture and only sketch the proof. Consider first the parametric form of (1.10a):

$$\begin{aligned} \dot{Y}_\beta &= -\beta X_\beta(1 - Y_\beta^2) \\ \dot{X}_\beta &= Y_\beta - X_\beta \end{aligned} \tag{2.1}$$

The phase portrait of Eq. (2.1) is given in Fig. 1. Taking $w_\beta(\cdot) = \tanh^{-1}[y_\beta(\cdot)]$ in Eq. (2.1), one obtains

$$\ddot{w}_\beta + \dot{w}_\beta + \beta \tanh w_\beta = 0 \tag{2.2}$$

describing the damped motion of the classical particle in the potential well $V_\beta(w) = \beta \log \cosh w$. The function y_β corresponds to the solution of (2.2) with $w_\beta(-\infty) = \infty$ and $\dot{w}_\beta(-\infty) = -\beta$ (which is the terminal velocity in (2.2)). The point $w = 0$ is a stable equilibrium point. In a neighborhood of the origin, $\log \cosh w \approx w^2/2$ and Eq. (2.2) approaches the equation of the damped oscillator. The latter is over (under)damped for $\beta < \beta_c (\beta > \beta_c)$ and critically damped for

$$\beta_c = 1/4$$

Briefly, a proof is obtained as follows.

1. First, it is shown in the usual way that $(1, 1)$ is a saddle point of Eq. (2.1) for all $\beta \in \mathbb{R}^+$ while $(0, 0)$ is a stable node for $\beta \in (0, \beta_c]$ and a stable focus for $\beta > \beta_c$. As is known in the two-dimensional autonomous case, the trajectories approach a stable node along one of the critical directions $\{$which are given by $y = x/2 \cdot [1 \pm (1 - 4\beta)^{1/2}]$ in our case$\}$.

2. In a neighborhood in $[-1, 1]^2$ of $(1, 1)$, there exists a unique solution of (2.1) tangent to the unstable direction

$$y = (x - 1)(1 + 2\beta) + 1 \tag{2.3}$$

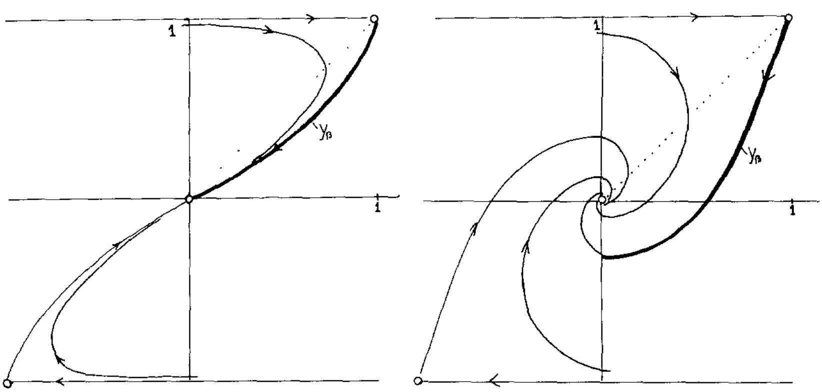

Fig. 1. (a) Phase portrait of Eq. (2.1) for $\beta \leqslant \beta_c$. (b) Phase portrait of Eq. (2.1) for $\beta > \beta_c$.

This follows from the behavior of trajectories near a saddle point[6] (the existence of such a solution also follows from the proof of Lemma 3.1 given in the Appendix).

The second trajectory approaching (1, 1) is a singular one, with $Y_\beta(t) \equiv 1$; this trajectory gives the stable direction at (1, 1).

3. The total energy of the particle in (2.2) gives a Lyapunov function for Eq. (2.1). The Lyapunov function is used to prove that the solution obtained in step 2 extends to the whole real axis and also that, for this solution,

$$(X_\beta(t), Y_\beta(t)) \in (-1, 1)^2 \quad \text{for} \quad t \in \mathbb{R} \tag{2.4}$$

$$(X_\beta(t), Y_\beta(t)) \to (0, 0) \quad \text{for} \quad t \to \infty \tag{2.5}$$

To summarize these intermediate results, we have the following.

Lemma 2.2. For each $\beta \in \mathbb{R}^+$ the system (2.1) has a unique solution $(X_\beta(\cdot), Y_\beta(\cdot)): \mathbb{R} \mapsto (-1, 1)^2$ for which the path is tangent to the line (2.3) at $x = y = 1$.

Computing the field direction across the line $y = x$, one can show by the previous lemma that

$$X_\beta(t) > Y_\beta(t) \quad \text{for} \quad t \in (-\infty, t_\beta) \tag{2.6}$$

where $t_\beta \equiv \inf\{t: X_\beta(t) = 0\}$ for $\beta > \beta_c$ and $t_\beta \equiv \infty$ for $\beta \leqslant \beta_c$. The function y_β is now defined by its graph

$$y_\beta = \overline{\{(X_\beta(t), Y_\beta(t)): t \in (-\infty, t_\beta)\}} \tag{2.7}$$

where $\bar{A}$ denotes the closure of the set A.

It is easy to see that this is a correct definition and that y_β satisfies (1.10a)–(1.10c). In view of step 2, y_β is the only solution of (1.10a)–(1.10c); the regularity of the obtained y_β is obvious. ∎

Lemma 2.3. (i) $y'_\beta(x) > 0$ for $x \in (0, 1)$ and $\beta \in \mathbb{R}^+$. Also,

$$y_\beta(x) > x/2 \cdot [1 + (1 - 4\beta)^{1/2}] \quad \text{for} \quad (x, \beta) \in (0, 1) \times (0, \beta_c] \tag{2.8}$$

$$y_\beta(0) = 0 \quad \text{for} \quad \beta \in (0, \beta_c] \quad \text{and} \quad y_\beta(0) \in (-1, 0) \quad \text{for} \quad \beta > \beta_c \tag{2.9}$$

(ii) y'_β extends to a $C[0, 1]$ map. We have $y'_\beta(1) = 1 + 2\beta$ for $\beta \in \mathbb{R}^+$;

$$y'_\beta(0) = 1/2 \cdot [1 + (1 - 4\beta)^{1/2}] \quad \text{for} \quad \beta \in (0, \beta_c]$$

$$y'_\beta(0) = 0 \quad \text{for} \quad \beta > \beta_c$$

(iii) y_β converges uniformly, as $\beta \to 0$, to the function $y_0(x) = x$, the singular solution for $\beta = 0$ of the equation $y'_\beta(y_\beta - x) = \beta x(1 - y_\beta^2)$.

If (2.17) holds, then, taking $\phi_n(k) = f_n(k)/\tilde{f}_n(k)$, we get

$$\phi_{n+1}(k) = \phi_n(k+1)\exp\left(-\beta\frac{k}{n+1}\right)\cdot\frac{\tilde{f}_n(k+1)}{\tilde{f}_{n+1}(k)} + \phi_n(k-1)\exp\left(\beta\frac{k}{n+1}\right)\cdot\frac{\tilde{f}_n(k-1)}{\tilde{f}_{n+1}(k)}$$

from which it follows inductively that

$$\text{const}\cdot\exp\left(-\sum_{j=1}^{n}\varepsilon'_j\right) < \phi_{n+1}(k) < \text{const}'\cdot\exp\left(\sum_{j=1}^{n}\varepsilon'_j\right), \quad k\in S_{n+1}, \ n\in\mathbb{N}$$

implying (2.14).

Remark. The second factor in (2.16) [which obviously does not contribute to (2.14)] was added in order to allow the inductive approach to the problem. It originates in the fact that, for $\beta \geqslant \beta_c$, the limit probability distribution is actually a superposition of two states, corresponding to opposite values of the magnetization.

For the proof of (2.17) it suffices to consider that n is large enough and $k \geqslant 0$. Define for $0 \leqslant k \leqslant n-1$

$$\chi^{\pm}_{n+1}(k) = \exp\left[\mp\beta\frac{k}{n+1} + nG_0\left(\frac{|k\pm 1|}{n};\beta\right) - (n+1)\,G_0\left(\frac{k}{n+1};\beta\right)\right]$$

Also, let $\chi^{-}_{n+1}(n+1)$ be given by the above expression and $\chi^{+}_{n+1}(n+1) = 0$.

1. First we show that, for some $\{\varepsilon''_n\}_{n\in\mathbb{N}}$, with $\varepsilon''_n \to 0$ as $n\to\infty$,

$$|\log[\chi^{+}_{n+1}(k) + \chi^{-}_{n+1}(k)]| < \varepsilon''_n \qquad \text{for } 0 < k \in S_{n+1}, \quad n\in\mathbb{N}$$

For $0 < k \leqslant n-1$,

$$\chi^{\pm}_{n+1}(k) = \exp\left[\mp\beta\frac{k}{n+1} + \left(\frac{k}{n+1}\pm 1\right)G'_0\left(\frac{k}{n+1};\beta\right) - G_0\left(\frac{k}{n+1};\beta\right) + \frac{1}{2n}\left(\frac{k}{n+1}\pm 1\right)^2 G''_0(\xi_\pm;\beta)\right]$$

with

$$\xi_\pm \in \left(\frac{k-1}{n}, \frac{k+1}{n}\right)$$

It is easy to see that $G''_0(x;\beta) \leqslant \beta$ for $x\in(0,1)$. Then, by (1.9a) [observing also that $\chi^{+}_{n+1}(n+1) + \chi^{-}_{n+1}(n+1) = 1$],

$$\log[\chi^{+}_{n+1}(k) + \chi^{-}_{n+1}(k)] \leqslant 2\beta n^{-1} \qquad \text{for} \quad 0 < k \in S_{n+1}$$

For a lower bound, let $A(\beta)$ be such that for all $x \in (0, 1)$, $|G_0''(x; \beta)| \leqslant A(\beta)(1-x)^{-1}$ [see (2.12)]. For $0 < k \leqslant n - \log n$, we get

$$\log[\chi_{n+1}^{+}(k) + \chi_{n+1}^{-}(k)] \geqslant -\mathrm{const} \cdot A(\beta)/\log n$$

Then, for $n - \log n < k < n$, noting that by (2.12), $G_0(x; \beta)$ is decreasing for x close to 1, we obtain

$$\begin{aligned}
&\log[\chi_{n+1}^{+}(k) + \chi_{n+1}^{-}(k)] \\
&\qquad \geqslant \log \chi_{n+1}^{-}(k) \\
&\qquad \geqslant \beta(n - \log n)/(n+1) - G_0((n - \log n)/(n+1); \beta) \to 0 \qquad \text{as} \quad n \to \infty
\end{aligned}$$

since $G_0(\cdot\,; \beta)$ is continuous.

2. For $k = 0$,

$$\begin{aligned}
T_{n+1}(0) = \{&\exp[G_0'(0; \beta) - G_0(0; \beta)] \\
&+ \exp[-G_0'(0; \beta) - G_0(0; \beta)]\} \exp\left[\frac{1}{2n} G_0''(\xi; \beta)\right]
\end{aligned}$$

where $\xi \in (0, 1/n)$; (2.17) is obtained from (2.13) with $\varepsilon_n' = \mathrm{const}(\beta)/n$. For $0 < k \leqslant \log n$, we write

$$\frac{T_{n+1}(k)}{\chi_{n+1}^{+}(k) + \chi_{n+1}^{-}(k)} = (1 + e^{-\alpha k})^{-1} \left\{1 + e^{-\alpha k} \frac{\chi_{n+1}^{+}(k)\, e^{-\alpha} + \chi_{n+1}^{-}(k)\, e^{\alpha}}{\chi_{n+1}^{+}(k) + \chi_{n+1}^{-}(k)}\right\} \tag{2.18}$$

Note that

$$\chi_{n+1}^{\pm}(k) \exp(\mp\alpha) = \chi_{n+1}^{\mp}(k)\, \chi_{n+1}^{0}(k)$$

with

$$\chi_{n+1}^{0}(k) = \exp\left[\mp 2\beta \frac{k}{n+1} \pm 2\xi G_0''(\zeta; \beta)\right] \tag{2.19}$$

$$\xi \in \left(\frac{k-1}{n}, \frac{k+1}{n}\right), \qquad \zeta \in (0, \xi)$$

Regarding the expression obtained after the substitution of (2.19) into (2.18) as a weighted mean of χ_{n+1}^{0}, one obtains an estimation for ε_n', in this region, of the form $\mathrm{const}(\beta)/\log n$.

Finally, for $k \in (\log n, n+1]$ it follows directly from (2.18) that

$$|\log\{[\chi_{n+1}^{+}(k) + \chi_{n+1}^{-}(k)]^{-1}\, T_{n+1}(k)\}| < \mathrm{const} \cdot n^{-\alpha}$$

On account of step 1 above, this completes the proof of (i).

APPENDIX

Proof of Lemma 3.1. Note first that the only problem is the analyticity near the singular point (1, 1). It is sufficient to prove analyticity on $(x_0, 1) \times \mathbb{R}^+$ for some x_0, since on (0, 1), $y_\beta(x) < x$ and the solutions of (1.10) depend analytically on (β, x) and the initial conditions in any domain of regularity of the rhs of (1.10). In order to obtain the analyticity near the critical point, it is convenient to consider y_β as a perturbation of its asymptotic expression for $x \to 1$, which behaves analytically. With $x_\beta \equiv y_\beta^{-1}$, define $\xi_\beta : (0, 1) \mapsto \mathbb{R}$ by

$$\xi_\beta(w) = [1 - x_\beta(\tanh w)] \exp w \tag{A1}$$

By (1.10),

$$\frac{d\xi_\beta}{dw} = (1 + \beta^{-1})\, \xi_\beta - \frac{(1 - \tanh w)\, e^w - \xi_\beta^2 e^{-w}}{\beta(1 - \xi_\beta e^{-w})} \tag{A2a}$$

and

$$\lim_{w \to \infty} \xi_\beta(w) = 0 \tag{A2b}$$

In view of the definition of ξ_β, it is easy to see that Eqs. (A.2a) and (A.2b) have a unique solution and also that the needed analyticity of y_β is equivalent to that of ξ_β on $(w_0, \infty) \times \mathbb{R}^+$ for some w_0. The second term in the rhs of Eq. (A.2a) vanishes for $w \to \infty$. It is then natural to consider the following integral version of Eqs. (A.2a) and (A.2b):

$$\xi_\beta(w) = \beta^{-1} e^{w(\beta^{-1}+1)} \int_w^\infty ds\, e^{-s(\beta^{-1}+1)} \frac{(1 - \tanh s)\, e^s - \xi_\beta^2(s)\, e^{-s}}{1 - \xi_\beta(s)\, e^{-s}} \tag{A3}$$

We treat (A.3) as a fixed-point problem in a suitable analytic function space.

Let $\varepsilon > 0$, $\delta \in (0, 1)$. For $\alpha = \alpha(\varepsilon, \delta)$ large enough, let $S_{\alpha,\varepsilon}$ be the space of complex analytic bounded functions on the domain $D_{\alpha,\varepsilon} = \{(w, \beta) \in \mathbb{C}^2$: $\mathrm{Re}(w) > \alpha$, $\mathrm{Re}(\beta) > \varepsilon\}$. Consider the closed ball $B_\delta = \{f \in S_{\alpha,\varepsilon} : \|f\| \leqslant \delta\}$, where $\|\cdot\|$ denotes the sup norm on $D_{\alpha,\varepsilon}$. Define the nonlinear operator T on B_δ by

$$\begin{aligned}(Tg)(w, \beta)\\ &= \beta^{-1} e^{w(1+\beta^{-1})} \int_w^{\infty + i\,\mathrm{Im}(w)} d\tau\, e^{-(1+\beta^{-1})\tau} \frac{(1 - \tanh \tau)\, e^\tau - g^2(\tau, \beta)\, e^{-\tau}}{1 - g(\tau, \beta)\, e^{-\tau}}\end{aligned} \tag{A4}$$

The above integral is absolutely convergent for $g \in B_\delta$ and thus, by Hartog's theorem, $(Tg)(\cdot, \cdot)$ is an analytic function in the couple (w, β) on $D_{\alpha,\varepsilon}$. It can be checked in a straightforward manner that, for large enough α, $T(B_\delta) \subset B_\delta$ and also that T is a contraction on B_δ.

Let then $\tilde{\xi}_\beta$ be the unique fixed point of T in B_δ. It can be seen from (A.4) that T leaves invariant the closed subset $R_\delta \subset B_\delta$ of the functions that take real values for real β and w. Thus, $\tilde{\xi}_\beta \in R_\delta$ and since it satisfies (A.2a) and (A.2b) for (real) $\beta > \varepsilon$ and $w > \alpha$, then $\tilde{\xi}_\beta(w) = \xi_\beta(w)$ for $(w, \beta) \in (\alpha, \infty) \times (\varepsilon, \infty)$. Therefore y_β is real analytic on $(0, 1) \times (\varepsilon, \infty)$. Since ε is arbitrary, this concludes the proof. ∎

Proof of Lemma 3.2. In the following we shall use some short-hand notations such as $y_\beta(g)$ for $y_\beta \circ g_\beta^{-1}(g)$ or $y_\beta(h)$ for $y_\beta(m(\beta, h))$.

The key to the approach is the use of the projective variable $g = y/x$ in a kind of σ-process in order to compare y_β with the solutions $\tilde{y}_\beta$ of the linearized equation in a neighborhood of $x = 0$. The values of interest for g are between $g_0 = \tanh \beta_c < \inf_{\beta > \beta_c} g_\beta(h = 0)$ and some $g_1 > y'_{\beta_c}(x = 0)$, say $g_1 = 1/2 + 1/8$.

Define the function $\tilde{y}_\beta$ by

$$\tilde{y}_\beta(g) = [(g - 1/2)^2 + \omega^2]^{-1/2} \exp\left\{\frac{1}{2\omega}\left[\arctan\left(\frac{g - 1/2}{\omega}\right) - \arctan\left(\frac{g_1 - 1/2}{\omega}\right)\right]\right\} \tag{A5}$$

with $\omega = (\beta - \beta_c)^{1/2}$. Consider $\beta_0 > \beta_c$. For (i) we prove a stronger result, namely:

(i′) For all $g \in (g_0, g_1)$ and $\beta \in (\beta_c, \beta_0]$,

$$K_1 < y_\beta(g)/\tilde{y}_\beta(g) < K_2 \tag{A6}$$

where $K_{1,2} \in \mathbb{R}^+$ are β independent (but may depend on g_0, g_1, or β_0).

Proof. By (1.10)

$$\log[y_\beta(1 - y_\beta^2)^{-1/2}]\big|_{y_\beta(g)}^{y_\beta(g_1)} = \beta \int_g^{g_1} \frac{d\gamma}{\gamma \tilde{\Gamma}_\beta(\gamma)} + \beta \int_g^{g_1} d\gamma\, E_\beta(\gamma) \tag{A7}$$

with $\tilde{\Gamma}_\beta(g) = (g - 1/2)^2 + \omega^2$ and $E_\beta(g) = \beta y_\beta^2(g)[g\Gamma_\beta(g)\,\tilde{\Gamma}_\beta(g)]^{-1}$ [see (2.11)]. Noting that $y_\beta(g) < g_1$, the upper bound in (A.6) follows from the positivity of E_β. For the reverse inequality, we show, using an estimation of E_β, that

$$\int_g^{g_1} d\gamma\, E_\beta(\gamma) < \text{const} \qquad \text{for} \quad (\beta, g) \in (\beta_c, \beta_0] \times (g_0, g_1)$$

Let

$$\tilde{E}_\beta(g) = \frac{d}{dg} \log \frac{\tilde{\Gamma}_\beta(g)}{\Gamma_\beta(g)}$$

Since $\tilde{\Gamma}_\beta < \Gamma_\beta$ and $y_\beta(g) < g$, it follows in a direct manner that $\tilde{E}_\beta(g) > E_\beta(g)/8$ for $(g, \beta) \in (0, g_1) \times (\beta_c, \beta_0]$. Now,

$$\int_g^{g_1} d\gamma\, \tilde{E}_\beta(\gamma) < \log[\tilde{\Gamma}_\beta(g_1)/\Gamma_\beta(g)] < \text{const}$$

The rightmost inequality follows for $\beta \geqslant \beta_c$ from the continuity of $y_\beta(g_1)$ in β and the positivity of $\Gamma_\beta(g_1)$ [these, in turn, result from Lemma 2.3(iv)].

Let $h_0 \equiv \inf\{h: g_\beta(h) = g_1\}$; $h_0 > 0$ (by a compactness argument). Now, inequality (A.6) works in terms of $y_\beta(h)$ for $h \in [0, h_0]$, proving (i). For the proof of (ii), we shall indicate the main steps, omitting the lengthy details. Define

$$z_\beta(h) = F(y_\beta(h)) \qquad \text{with } F(t) = -\tfrac{1}{2}\log(1 - t^2)$$

z_β satisfies the more convenient equation

$$\frac{dz_\beta}{dh} = (\log\cosh)^{-1} z_\beta - \beta h \tag{A8}$$

We shall use the equations in variations for z_β in order to estimate $(\partial^k/\partial\beta^k)\, z_\beta(h)$, $k \in \mathbb{N}$,

$$\frac{d}{dh}\frac{\partial^k z_\beta}{\partial\beta^k} = \frac{1}{y_\beta}\frac{\partial^k z_\beta}{\partial\beta^k} + Z_{\beta,k} \tag{A9}$$

with

$$Z_{\beta,1}(h) = -h$$

and, for $k > 1$,

$$Z_{\beta,k}(h) = \frac{\partial^k}{\partial\beta^k}[(\log\cosh)^{-1} z_\beta(h)] - \frac{1}{y_\beta(h)}\frac{\partial^k}{\partial\beta^k} z_\beta(h)$$

The initial conditions for (A.9), $(\partial^k/\partial\beta^k)\, z_\beta(h_0)$ are, by Lemma 3.1 and the positivity of $y_\beta(h_0)$, real analytic functions in $\beta \in \mathbb{R}^+$. More generally, let ψ_β be the solution in $[0, h_0]$ of the equation

$$\frac{d}{dh}\psi_\beta = \frac{1}{y_\beta}\psi_\beta + \Lambda_\beta \qquad \text{with} \quad \psi_\beta(h_0) = \psi_0(\beta) \tag{A10}$$

where $\psi_0(\beta)$ is continuous in $\beta \in [\beta_c, \beta_0]$ and $(h, \beta) \mapsto \Lambda_\beta(h)$ is continuous on $D_{\beta h} \equiv [0, h_0] \times (\beta_c, \beta_0]$. Assume further that for some const > 0,

$$|\Lambda_\beta(h)| < \text{const} \cdot y_\beta(h)$$

Then we have the following result.

Lemma A.1. For β_0 close enough to β_c we have, with a β-independent constant,

$$|\psi_\beta(h)| < \text{const} \cdot \omega^{-3} y_\beta^2(h) \qquad \text{in} \quad D_{\beta h} \tag{A11}$$

Proof. The solution of (A.10) is

$$\begin{aligned}\psi_\beta(h) = \psi_0(\beta) \exp\left[-\int_h^{h_0} dh' \frac{1}{y_\beta(h')}\right] - \int_h^{h_0} dh'\, \Lambda_\beta(h') \\ \times \exp\left[-\int_h^{h'} dh'' \frac{1}{y_\beta(h'')}\right]\end{aligned} \tag{A12}$$

Now,

$$\int_h^{h'} dh'' \frac{1}{y_\beta(h'')} = \int_{g_\beta(h)}^{g_\beta(h')} dg \frac{1}{\Gamma_\beta(g)}$$

Reasoning as in the proof of (i') and using the results obtained there, it can be verified that

$$\int_h^{h_0} dh'' \frac{1}{y_\beta(h'')} > \log \frac{\text{const}}{y_\beta^2(h)}$$

and also that

$$\begin{aligned}&\left|\int_h^{h_0} dh'\, \Lambda_\beta(h') \exp\left[-\int_h^{h'} dh'' \frac{1}{y_\beta(h'')}\right]\right| \\ &\qquad < \text{const} \cdot \omega^{-2} y_\beta^2(h) \log \frac{1}{y_\beta(h)} \\ &\qquad < \text{const} \cdot \omega^{-3} y_\beta^2(h) \quad \blacksquare\end{aligned}$$

As a consequence, for some sequences $\{\beta_n'\}_{n \in \mathbb{N}}$ in $(\beta_c, \beta_0]$ and $\{p_n'\}_{n \in \mathbb{N}}$ in $\mathbb{R}^+$,

$$\left|\frac{\partial^n}{\partial \beta^n} z_\beta(h)\right| < \text{const}_n \cdot \omega^{-p_n'} y_\beta^2(h) \quad \text{for } (h, \beta) \in D_{\beta h}, \quad n \in \mathbb{N} \tag{A13}$$

(A.13) is proved inductively, noting that $Z_{\beta,k}$ depends only on the $(\partial^j/\partial\beta^j)\, z_\beta$ with $j \leqslant k-1$ and using Lemma A.1 and the real analyticity of $t^{-1/2}\log\cosh t$ for the needed estimations. Now relation (3.6) follows in an obvious way. ▮

Remark. The factor ω^{-3} in Lemma A.1 is optimal. This can be used to show that the derivatives in β of y_β and $\tilde{y}_\beta$ for $h = h_0$ have the same asymptotic behavior.

ACKNOWLEDGMENTS

The authors wish to thank Dr. M. Bundaru and Dr. O. T. Popp for useful discussions and suggestions. We are particularly indebted to Dr. N. Angelescu, who read the manuscript and suggested many improvements.

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